Eur. Phys. J. B **19**, 133–138 (2001)

**THE EUROPEAN PHYSICAL JOURNAL B**

EDP Sciences


# Multiscaling and multifractality in an one-dimensional Ising model

W. Jeżewski[a]

Institute of Molecular Physics, Polish Academy of Sciences, Smoluchowskiego 17/19, 60-179 Poznań, Poland

Received 10 July 2000 and Received in final form 6 November 2000

**Abstract.** Scaling properties of the Gibbs distribution of a finite-size one-dimensional Ising model are investigated as the thermodynamic limit is approached. It is shown that, for each nonzero temperature, coarse-grained probabilities of the appearance of particular energy levels display multiscaling with the scaling length $\ell = 1/M_n$, where $n$ denotes the number of spins and $M_n$ is the total number of energy levels. Using the multifractal formalism, the probabilities are argued to reveal also multifractal properties.

**PACS.** 05.50.+q Lattice theory and statistics (Ising, Potts, etc.) – 05.70.-a Thermodynamics – 64.10.+h General theory of equations of state and phase equilibria – 68.35.Rh Phase transitions and critical phenomena

## 1 Introduction

The structure of complex objects is usually studied by means of statistical methods involving coarse-grained probability measures assigned to pieces of the objects [1–3]. These measures can exhibit singularities as the length scale of the piece size tends to zero. The underlying scaling behavior of the measures can display a simple form, when the singularity strength (the Hölder exponent) possesses a single value, or can reveal complicated properties, when the singularities run over continuous spectra. In the latter case, complex scaling properties of the probability measures are called multiscaling [4,5]. As well known, a given measure characterized by a spectrum of singularities can be divided into submeasures, each associated with a particular value of the singularity exponent and each supported, in general, by a fractal set. If singularities characterizing such subsupports and their dimensions spread continuously over some ranges, then the support of the whole probability measure form the so-called multifractal. It should be pointed out, however, that multiscaling does not always imply multifractality [5].

The probabilistic approach to characterize complicate objects by studying scaling properties of singular measures associated with these objects has widely been used, no matter of their origin, and irrespective of the way of defining or generating their coarse-grained measures. In particular, the approach has been applied to analyze the scaling behavior of temperature-dependent probability measures (Gibbs distributions) of finite-size, two-dimensional Ising systems [6]. Clearly, these probability measures are supported by discrete spectra of energy levels. For a given finite-size system of $n$ spins, the number of energy levels is $M_n \sim n$. Accordingly, the support of the probability measure determined for a system of $n$ spins can be covered by $M_n$ segments, and the length scale characterizing the measure can be assumed to be $\ell \sim 1/M_n$ [6]. The resulting scaling exponents are dependent on the temperature variable. It has been shown that Gibbs distributions of two-dimensional systems reveal specific multifractal structures, and that the structures are related to some thermodynamic properties of these systems [6]. However, calculations of singularity spectra for Gibbs distributions involve rates of energy degeneracies and, thereby, in cases of two-dimensional (and higher-dimensional) Ising models, the calculations can be performed only for rather small systems. Thus, in order to investigate the question of multiscaling properties of Gibbs distributions of spin systems more thoroughly, one must resort to a more simple model, for which degeneracy rates can exactly be determined.

Here, scaling properties of the Gibbs distribution of a one-dimensional Ising system are examined. For this system, degeneracies of energy levels are exactly known for each $n$, and thereof scaling properties of the probability measure can be studied systematically for very large $n$. Consequently, it will be exactly shown the existence of the phenomenon of multiscaling in the system. An argumentation for the occurrence of the multifractality will also be presented.

## 2 Singularities of the Gibbs distribution

Consider a zero-field Ising system on a one-dimensional lattice with periodic boundary conditions. The spin variables $s_i = \pm 1$, $i = 1, 2, ..., n$, are taken to be coupled

[a] e-mail: jezewski@ifmpan.poznan.pl

by nearest-neighbor interactions $J$. For simplicity, the total number of spins will be assumed to take only even values. The reduced energy spectrum of the system of $n$ spins is then given by

$$E_i^{(n)}(K) = (n-4i)K \ , \ i = 0, 1, ..., M_n - 1, \quad (1)$$

with the reduced interaction $K = J/k_{\mathrm{B}}T$, where $T$ denotes the temperature, and with

$$M_n = \frac{n}{2} + 1 \quad (2)$$

being the total number of energy levels. The normalized probability that the system occupies the $i$th energy level is determined by

$$p_i^{(n)}(K) = \frac{2}{Z_n(K)}\binom{n}{2i}\mathrm{e}^{(n-4i)K} \quad (3)$$

with the partition function

$$Z_n(K) = 2\sum_{i=0}^{M_n-1}\binom{n}{2i}\mathrm{e}^{(n-4i)K} = (2\cosh K)^n + (2\sinh K)^n. \quad (4)$$

It follows from (2) and (3) that, for energy levels of sufficiently low degeneracies, the respective probabilities decay as powers of $\mathrm{e}^{-n}$. However, for the energy levels of high degeneracies, the probabilities can be expected to decay more slowly. This can easily be seen by using the Stirling approximation for the degeneracy factor in (3). Then, setting

$$2i = r_i n \ , \ i = 0, 1, ..., M_n - 1, \quad (5)$$

with the rational numbers $0 \leq r_i \leq 1$, one has

$$\ln[p_i^{(n)}(K)] = nS(K, r_i) - \frac{1}{2}\{\ln n + \ln[r_i(1-r_i)] - \ln(2/\pi)\}, \quad (6)$$

where

$$S(K, r_i) = (1-2r_i)K - h_n(K) - r_i \ln r_i - (1-r_i)\ln(1-r_i) \quad (7)$$

with

$$h_n(K) = \ln(2\cosh K) + \frac{1}{n}\ln[1 + (\tanh K)^n]. \quad (8)$$

Representing the rational numbers $r_i$ for each $K$ by

$$r_i = a(K) + b_i(K)\sqrt{\frac{\ln n}{n}} + g_i(K, n), \quad i = 0, 1, ..., M_n - 1, \quad (9)$$

where $a(K)$, $b_i(K)$, $g_i(K, n)$ are, in general, irrational numbers, such that

$$a(K) = \frac{1}{\mathrm{e}^{2K} + 1}, \quad (10)$$

$b_i(K)$ are finite, independent of $n$, and

$$g_i(K, n)/\sqrt{\frac{\ln n}{n}} \to 0 \quad (11)$$

as $n \to \infty$, yields

$$S(K, a) = \frac{1}{n}\ln[1 + (\tanh K)^n]. \quad (12)$$

Hence, one derives

$$\ln[p_i^{(n)}(K)] = -\alpha_i(K)\ln n + \mathcal{O}(\varepsilon(K, n)), \quad (13)$$

where

$$\alpha_i(K) = \frac{1}{2}\left(1 + 4b_i^2(K)(\cosh K)^2 + \frac{\ln[r_i(1-r_i)]}{\ln n}\right), \quad (14)$$

and $\varepsilon(K, n)$ is the largest of terms of orders $\sqrt{\frac{\ln n}{n}}$ and $g_i(K, n)\sqrt{n \ln n}$. Using (3) and (13), one can easily verify that the minimum value of $\alpha_i(K)$ is given for $K = 0$ by $\alpha_m(0) = \frac{1}{2}$, where $m = n/4$ if $n = 4j$ (with $j$ taking on positive integer values), while $m = (n+2)/4$ when $n = 4j + 2$. Since $r_m = \frac{1}{2}$ for $n \to \infty$ (see Eq. (5)), the relation (14) implies that $b_m(0) = 0$ in the thermodynamic limit. However, for sufficiently large $K$, the minimal value of $\alpha_i(K)$ is associated with $i \ll M_n$. According to (3) and (13), in the limits $n \to \infty$ and $K \to \infty$, this minimal value is $\alpha_0(\infty) = 0$. Since for $i \ll M_n$ $r_i$ is of the order of $1/n$, it follows from equation (14) that $\lim_{K\to\infty} b_0(K)\mathrm{e}^K = 0$.

Thus, as $n \to \infty$, the probabilities $p_i^{(n)}(K)$ satisfy for each finite $K$ the scaling law

$$p_i^{(n)}(K) \sim \left(\frac{1}{M_n}\right)^{\alpha_i(K)} \quad (15)$$

with the scaling length being assumed to be $\ell_n \sim 1/M_n$ rather $\ell_n \sim 1/n$, because $M_n$ is (for each $n$) the total number of elements of the support of the probability measure. The discrete index $i$ in (15) is restricted to take those values which are determined by (5, 9, 10), and the condition (11). If the condition (11) is not satisfied for some $i$ and if $g_i(K, n)/\sqrt{\frac{\ln n}{n}} \to \infty$ as $n \to \infty$ (with $g_i(K, n)$ being finite), then, for each finite $K$, the exponents $\alpha_i(K)$ are dependent on $n$ and tend to infinity when $n$ grows, violating the scaling law (15). Similarly, the scaling law (15) is not fulfilled if the requirement (10) does not hold. Consequently, for $n \gg 1$, the largest length scale of the probabilities $p_i^{(n)}(K)$ is $\ell_n \sim 1/M_n$. Clearly, the smallest length scale is determined by $\ell_n \sim \mathrm{e}^{-n}$. Accordingly, all possible length scales that decay more rapidly than $\ell_n \sim 1/M_n$ as $n \to \infty$ emerge when $a(K)$ and/or $g_i(K, n)$ do not satisfy conditions (10) and (11), respectively. Note that, since $r_i$ are defined as belonging to a unit interval, all $g_i(K, n)$ must remain finite as $n \to \infty$.

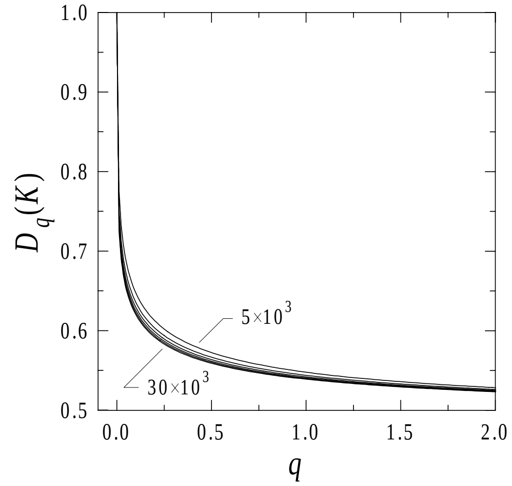


**Fig. 1.** The generalized dimension $D_q(K)$ *vs.* $q$ for $q \geq 0$, $K = 0.1$, and $n = 5 \times 10^3, 10 \times 10^3, ..., 30 \times 10^3$.

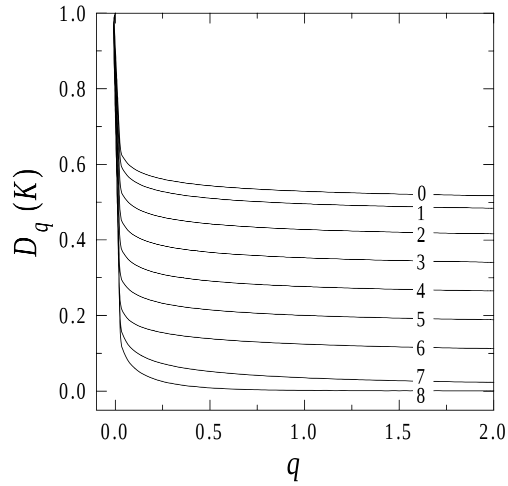


**Fig. 2.** $D_q(K)$ *vs.* $q \geq 0$ for $K = 0, 1, ..., 8$ and $n = 10^6$.

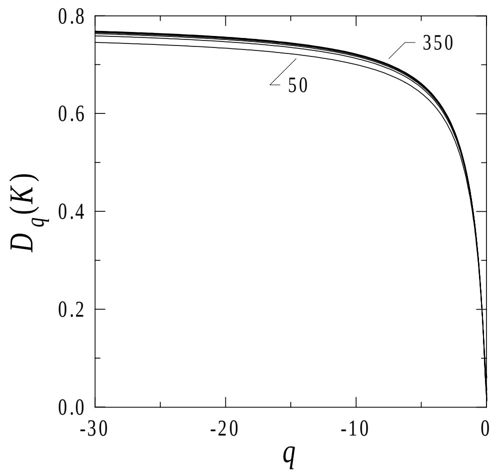


**Fig. 3.** The generalized dimension $\bar{D}_q(K)$ *vs.* $q < 0$ for $K = 0.1$ and $n = 50, 100, ..., 350$.

Detailed analysis of the scaling behavior of $p_i^{(n)}(K)$ can be performed in a standard way, by investigating properties of the *generalized* partition function (*cf.* [1,7,8])

$$\chi_n(q, K) = \sum_{i=1}^{M_n} [p_i^{(n)}(K)]^q \ , \tag{16}$$

where $q \in (-\infty, \infty)$. For $q \geq 0$, this function is expected to scale in the limit $n \to \infty$ as

$$\chi_n(q, K) \sim \left(\frac{1}{M_n}\right)^{(q-1)D_q(K)} \tag{17}$$

with the generalized dimensions $D_q$ being dependent on $K$. It should be pointed out, however, that $D_0(K) = 1$ for all $K$. This reflects the fact that, according to (1), the support of the entire probability measure is uniform for all $K$. The dependence of $D_q(K)$ on $q$ for $q \geq 0$, $K = 0.1$, and various $n$ is shown in Figure 1. It is seen that $D_q(K)$ converges rapidly as $n$ increases. In Figure 2 the dependence of $D_q(K)$ on $q \geq 0$ is presented for $n = 10^6$ and for diverse values of $K$. Note, that $D_q(0) \to \frac{1}{2}$ and $D_q(\infty) \to 0$ when $q$ grows. Since $D_\infty(K) = \min\{\alpha_i(K)\}$ [1], this is in agreement with the predictions, based on the relation (14), that $\min\{\alpha_i(0)\} = \frac{1}{2}$ and $\min\{\alpha_i(\infty)\} = 0$. The visible dependence of $D_q$ on $q$ provides a strong evidence of the multiscaling property [4] of the probabilities $p_i^{(n)}(K)$ associated with the largest length scale $\ell_n \sim 1/M_n$ and related to highly degenerated energy levels $E_i^{(n)}(K)$ with the index $i$ determined by (5, 9–11). Clearly, the continuous spectra $D_q(K)$ are connected in the limit $n \to \infty$ to continuous spectra of the Hölder exponents $\alpha(q, K)$ [1].

As mentioned above, the smallest length scale of the probabilities $p_i^{(n)}(K)$, corresponding to energies of low degeneracies, is given by $\ell_n \sim \mathrm{e}^{-n}$ rather than by $\ell_n \sim 1/M_n$. Hence, the partition function $\chi_n(q, K)$ is assumed to satisfy for $q < 0$ and $n \gg 1$ the scaling relation

$$\chi_n(q, K) \sim \mathrm{e}^{-n(q-1)\bar{D}_q(K)} \tag{18}$$

with the generalized dimensions $\bar{D}_q(K)$ defined for $q < 0$. The dependence of $\bar{D}_q(K)$ on $q$ illustrates Figure 3 for $K = 0.1$ and for various $n$. This figure displays the convergence of $\bar{D}_q(K)$ as $n$ is increased. The dependence of $\bar{D}_q(K)$ on $q$ for $n = 200$ and for varied values of $K$ is shown in Figure 4.

## 3 Multifractal behavior of the Gibbs distribution

This section discusses the question whether the multiscaling property of the probability measure $p_i^{(n)}(K)$ implies a multifractal structure of the support of the measure. As follows from (1), the coarse-grained support of this measure is spread over a discrete spectrum of energy levels, belonging to a range, which tends to infinity as the thermodynamic limit is approached. Thus, a possible multifractality would refer to growing fractal sets of a lower cutoff length scales (*cf.* [9]). Apparently, by rescaling the total range of energy levels to a constant interval

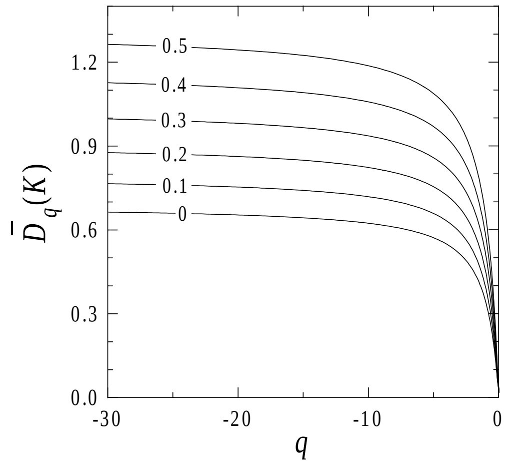


**Fig. 4.** $\bar{D}_q(K)$ *vs.* $q < 0$ for $K = 0, 0.1, ..., 0.5$ and $n = 200$.

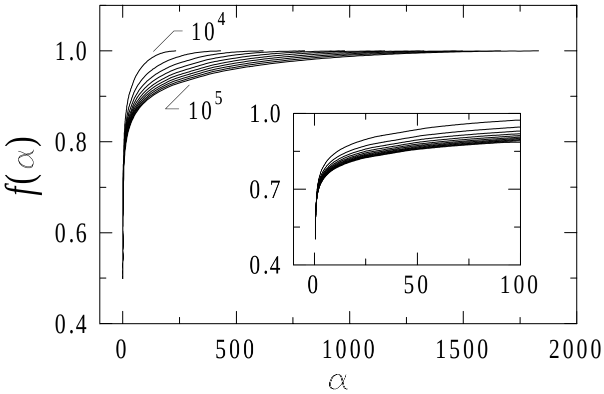


**Fig. 5.** The spectra $f(\alpha(q, K))$ for $K = 0.1$ and $n = 10^4, 2 \times 10^4, ..., 10 \times 10^4$. The inset displays a rather fast convergence of the singularity spectra for relatively small values of $\alpha(q, K)$ as $n$ grows.

(independent of $n$), one could obtain a true multifractal set (with zero lower cutoff length scale).

Essentially, the structure of the support of the measure $p_i^{(n)}(K)$ can be studied by determining the singularity spectrum $f(\alpha(q, K))$ of the Hölder exponents $\alpha(q, K)$ characterizing this measure [1,6]. Typically, in cases of multifractal sets, the $f(\alpha)$ spectra are continuous functions of $\alpha$. Clearly, in cases of monofractals, objects consisting of finite numbers of monofractals, or nonfractal sets, the singularity spectra should be discrete. It is well known, however, that standard multifractal methods [1,10] generate top envelopes of singularity spectra, missing possible interior points (which do not belong to the envelopes) [11]. Moreover, in cases of nonmultifractals or even in cases of nonfractal sets, these methods can produce spurious points forming continuous spectra [11]. Owing to the above shortcomings, the standard multifractal formalism cannot, in general, yield reliable qualitative conclusions concerning possible multifractal properties of considered measures. Nevertheless, using additionally special diagnostic procedures, one can identify spurious points, as well as one can detect hidden points [11]. This enable one to judge whether a true $f(\alpha)$ spectrum is in a given case continuous or not. Below, the $f(\alpha(q, K))$ spectra determined for the measure $p_i^{(n)}(K)$ by applying the steepest descent procedure [1] will be examined with the aid of a diagnostic procedure based on the sliding window fractal analysis [11].

The spectra $f(\alpha(q, K))$ derived for $q \geq 0$, $K = 0.1$, and for large numbers of spins are plotted in Figure 5. As one can see, the exponent $\alpha$ corresponding to $q = 0$ grows as n increases. It proves that, for $n = \infty$ and for each finite $K$, $\alpha(0, K) = \infty$ and, according to the multifractal analysis [1], $\frac{\partial}{\partial q} D_q(K) \mid_{q=0} = -\infty$ in the thermodynamic limit. Obviously, the divergence of $\alpha(0, K)$ for growing $n$ reflects the presence of length scales that diminish more rapidly than $\ell_n \sim 1/M_n$ when $n$ increases. Numerical results obtained for very large $n$ and various $\alpha(q, K)$ indicate, however, that, the spectra $f(\alpha(q, K))$ with $q > 0$ are convergent for each finite $K$ when $n$ increases (see also Fig. 5). Evidently, the shape of these spectra resembles for each $K$ a typical form of left sided multifractal spectra [12,13]. It should be noted that, since $\alpha(0, K) = \infty$ in the thermodynamic limit, right sides of the spectra $f(\alpha(q, K))$ do not exist. Consequently, the smooth transition of each spectrum to the asymptotic value $f(\alpha(q, K)) = 1$ at $q = 0$ can be considered as an infinite-order transition [13].

The reliability of the spectra $f(\alpha(q, K))$ can be analyzed using the moving window procedure [11]. This procedure allows one to examine how the range of the Hölder exponents derived within a window (subsupport of the probability measure) of a given length varies as the center point of the window moves. For system under study, the windows are determined by the discrete index $i$ belonging to discrete intervals $[i - \frac{s}{2}, i + \frac{s}{2}]$, $i = \frac{s}{2}, \frac{s}{2} + 1, ..., \frac{n}{2} - \frac{s}{2}$, with the window length $1 < s < \frac{n}{2}$ being a positive integer. Clearly, the extreme Hölder exponents $\alpha_{\min}$ and $\alpha_{\max}$ determining widths of singularity spectra connected with successive windows can be derived directly from the scaling relation (15). The dependence of the extreme values of $\alpha$ on the normalized variable $x = (i - \frac{s}{2})/(\frac{n}{2} - s)$ for sliding windows of the length $s = 100$ and for systems with the same temperature parameter $K = 0.1$, but of different sizes $n = 5 \times 10^3$, $n = 10 \times 10^3$, $n = 15 \times 10^3$, is displayed in Figure 6. It follows that, for a given $K$, the extreme Hölder exponents $\alpha_{\min}$ and $\alpha_{\max}$ vary within the whole range of $x$. The dependence of these exponents on $x$ is especially fast near the end points $x = 0$ and $x = 1$. This is entirely understandable, since regions of $x$ near these end points correspond to low degenerated energy levels, for which the scaling law (15) does not hold. Clearly, both the extreme exponents take on for given $K$ and $n$ the smallest values in a region of $x$ associated with the most probable energy levels. For each value of $x$, except of those values that belong to a relatively narrow interval corresponding to the most probable energy levels, the indices $\alpha_{\min}$ and $\alpha_{\max}$ turn out to increase as $n$ grows. The length of the narrow interval of values of $x$ (at which $\alpha_{\min}$ and $\alpha_{\max}$ continue to decrease when $n$ grows) proves to diminish as $n$ increases. Consequently, a range of $x$ for which $\alpha_{\min} < \omega_1$

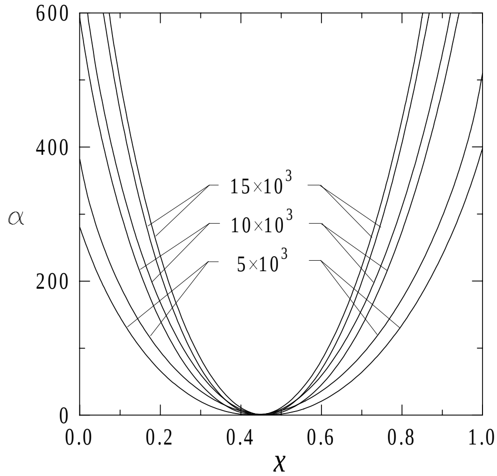


**Fig. 6.** The extreme exponents $\alpha_{\min}$ and $\alpha_{\max}$ as functions of normalized centers of sliding windows of the length $s = 100$, for $K = 0.1$ and $n = 5 \times 10^3, 10 \times 10^3, 15 \times 10^3$.

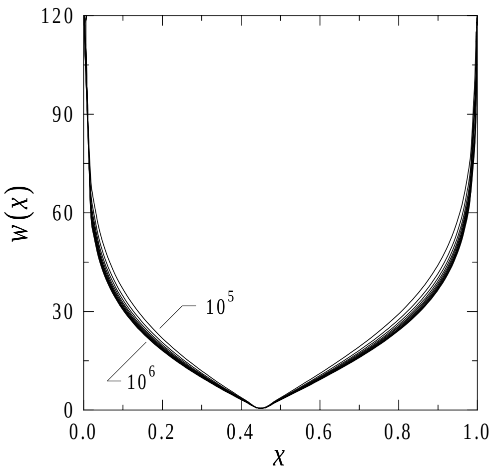


**Fig. 7.** The width function $w(x)$ for $s = 100$, $K = 0.1$ and $n = 10^5, 2 \times 10^5, ..., 10 \times 10^5$.

and $\alpha_{\max} < \omega_2$, where $\omega_1$ and $\omega_2$ are some constants (independent of $n$), shrinks (for fixed $K$ and $s$) when $n$ grows. However, the number of energy levels associated with this range of $x$ turns out to increase as $n \to \infty$.

It is also remarkable that, for $n$ growing to infinity, the width $w = \alpha_{\min} - \alpha_{\max}$ of the singularity spectrum remains nonzero for each $K$ and for each $x$, except for a one point $x = x_0(K)$ at which, however, the width vanishes only in the case of $n = \infty$. Furthermore, the width function $w(x)$ proves to be convergent when $n \to \infty$ (for each finite $K$ and for fixed $s$), as shown in Figure 7. Obviously, behaviors of the extreme Hölder exponents and the width of the $f(\alpha)$ spectrum as functions of $x$ are qualitatively similar for different values of $K$ and for various window lengths $s$. Generally, the extreme exponent $\alpha_{\max}$ and the width function take on for given $K$, $n$, and $x$ greater values for sliding windows of longer lengths. This is reflected in an untypical left-sided character of the spectra $f(\alpha(q, K))$ and is a consequence of the fact that the scaling law (15) is satisfied within a rather narrow range of rescaled center points $x$ of sliding windows. Since the width of the singularity spectra associated with particular windows is, in general, nonzero and both the corresponding extreme Hölder exponents are smooth functions of $x$ for various $s$ and for growing $n$, there is no reason to suspect that the left-sided spectra $f(\alpha(q, K))$ (with $q \geq 0$) contain spurious points or hidden interior points, which would be not captured by these spectra.

The spectra $f(\alpha(q, K))$ could certainly be considered as true multifractal spectra, if the support of the probability measure $p_i^{(n)}(K)$ would be composed of infinitely many intertwined Cantor sets, each of which being a support of a submeasure characterized by a definite Hölder exponent $\alpha(q, K)$. The corresponding submeasures might form intertwined structures, if probabilities of the appearance of energy levels that differ considerably from each other, would be associated with an identical Hölder exponent, *i.e.*, if nonequal, in general, probabilities $p_{i+v_i}^{(n)}(K)$ and $p_i^{(n)}(K)$ with $v_i \to \infty$ as $n \to \infty$ would be connected to the same exponent $\alpha$. According to (5, 9, 14), one obtains for the same values of $b_i(K)$, and hence for the same exponent $\alpha$, that $v_i = \frac{n}{2}[g_{i+v_i}(K, n) - g_i(K, n)]$ with $g_i(K, n)$ satisfying the condition (11). Assuming that $g_i(K, n) = c_i(K)\frac{\ln n}{n}$, where $c_i(K)$ is independent of $n$, one has $v_i = \frac{1}{2}(c_{i+v_i} - c_i) \ln n \to \infty$ as $n \to \infty$. Note that the probabilities $p_j^{(n)}(K)$ with $i < j < v_i$ can correspond to different values of $\alpha$, determined by different values of $b_j(K)$. This indicates that the probability measure, characterized by various values of the Hölder exponent, can really create intertwined structures. The way, in which the probabilities $p_i^{(n)}(K)$ identified with particular exponents $\alpha_i(K)$ are redefined as $n$ grows, is established by equations (5, 9–11). Obviously, the probabilities $p_i^{(n)}(K)$ corresponding to the same exponent $\alpha$ can be different. This can easily be seen using the relations (3–13). Then one derives

$$p_i^{(n)}(K) = u_i^{(n)}(K) \left( \frac{1}{M_n} \right)^{\alpha_i} \tag{19}$$

with

$$u_i^{(n)}(K) = 4 \cosh K \exp[-b_i(K) g_i(K, n) \sqrt{n \ln n}] \tag{20}$$

being, in general, different for the same value of $b_i(K)$ (for some set of the indices $i$). Thus, the examination of the probability measure $p_i^{(n)}(K)$ as well as the analysis of the structure of its support, especially the sliding window analysis of the resulting singularity spectra $f(\alpha(q, K))$, suggest that these spectra can be treated as left-sided multifractal ones.

## 4 Conclusions

The normalized probabilities of the occupation of highly degenerated energy levels of one-dimensional Ising system have been proved to reveal multiscaling property with the scaling length $\ell_n \sim 1/M_n$. The multiscaling has been shown to be related to multifractality with respect to the probability measure. Due to the existence of drastically different length scales characterizing this measure, especially due to the occurrence of the length scale $\ell_n \sim \mathrm{e}^{-n}$ corresponding to energy levels of low degeneracies, the resulting multifractal spectra $f(\alpha(q,K))$ display the left-sided shape with $0 < \alpha(\infty,K) < \infty$ and $\alpha(0,K) = \infty$ for all finite $K$. It should be pointed out that a more familiar form exhibit the spectra $f(\bar{\alpha}(q,K))$ with $\bar{\alpha}(q,K) = 1/\alpha(q,K)$ [6]. These spectra have a finite width for all $K$, but are not always upward convex functions of $\bar{\alpha}(q,K)$. The scaling features of coarse-grained probability measure studied here have a general character and, consequently, the multiscaling and the anomalous behavior of singularity spectra can be considered as typical properties of various systems with discrete energy spectra.

## References

1. Th.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, B.I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
2. U. Frisch, G. Parisi, in *Turbulence and Predictability in Geophysical Flows and Climate Dynamics, International School of Physics "Enrico Fermi", Course LXXXVII*, edited by M. Ghill, R. Benzi, G. Parisi (North-Holland, New York, 1985); G. Paladin, A. Vulpiani, Phys. Rep. **156**, 147 (1987); U. Frisch, *Turbulence. The Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
3. H. Fujisaka, R. Hagihara, M. Inoue, Phys. Rev. A **38**, 3680 (1988); H. Fujisaka, M. Inoue, Phys. Rev. A **39**, 1376 (1989).
4. P. Meakin, A. Coniglio, H.E. Stanley, T.A. Witten, Phys. Rev. A **34**, 3325 (1986); A. Coniglio, M. Zannetti, Physica D **38**, 37 (1989); J.-P. Eckmann, I. Procaccia, Phys. Lett. A **182**, 93 (1993); M.H. Jensen, Phys. Rev. Lett. **83**, 76 (1999).
5. M.H. Jensen, G. Paladin, A. Vulpiani, Phys. Rev. Lett. **67**, 208 (1991).
6. W. Jeżewski, Phys. Rev. B **57**, 10 240 (1998); W. Jeżewski, Phys. Lett. A **263**, 112 (1999); W. Jeżewski, Physica A **278**, 235 (2000).
7. H.G.E. Hentschel, I. Procaccia, Physica D **8**, 435 (1983).
8. C. Amitrano, A. Coniglio, F. di Liberto, Phys. Rev. Lett. **57**, 1016 (1986).
9. T. Vicsek, Physica A **168**, 490 (1990).
10. A. Chhabra, R.V. Jensen, Phys. Rev. Lett. **62**, 1327 (1989).
11. D. Veneziano, G.E. Moglen, R.L. Bras, Phys. Rev. E **52**, 1387 (1995).
12. M.C. Gutzwiller, B.B. Mandelbrot, Phys. Rev. Lett. **60**, 673 (1988); T. Bohr, P. Cvitanović, M.H. Jensen, Europhys. Lett. **6**, 445 (1988); B. Duplantier, Phys. Rev. Lett. **84**, 1363 (2000).
13. B.B. Mandelbrot, Physica A **168**, 168 (1990); B.B. Mandelbrot, C.J.G. Evertsz, in *Fractals and Disordered Systems*, edited by A. Bunde, S. Havlin (Springer, Berlin, 1996).